POWER FLOW FOR N-TERMINAL NETWORKS WITH RADIATION

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The electromagnetic power, defined as the flux of the Poynting vector, reduces to the algebraic sum of the terminal powers for \( n \)-terminal networks under restrictive conditions that forbade radiation. The aim of this paper is to show how the circuit concept of power can be extended to circuits with radiation. The extended approach is valid also for poorly or non-shielded circuits exposed to external disturbing electromagnetic fields.

1. INTRODUCTION

The electromagnetics of circuits treats the ensemble of idealisations in classical circuit theory [1]. Circuit theory operates with primary variables such as electric current intensities, tensions (voltages), charges of capacitors, and magnetic fluxes through inductors. It is natural to seek that the electric and magnetic stored energy, and electromagnetic power flow, be expressed in terms of these primary variables. This is the case for non-radiating circuits. For such circuits, the two concepts of power, the circuit and the field one are reconciled [2-3].

It is possible to use the retardation concept to show that circuits may radiate energy, but the exact amount of radiation cannot be obtained without a true field theory. Similarly, it is possible to take into account skin and proximity effects in conducting connections or coil windings, eddy currents in inductor/transformer cores, or charge relaxation in capacitor lossy dielectrics, by using the transient parameters concept [4]. The transient parameter theory generalizes the Kirchhoff laws in the form of integral equations with transient parameters as kernels.

However, if in the last case the electromagnetic power flowing out of the bounding surface of the circuit can be expressed solely in terms of currents and potentials, in the former radiative case, electric and magnetic field variables have to be included.

The aim of this paper is to show how the radiation of circuits can be accounted for in the power flow of the electromagnetic energy. The basic ideas

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reported in [5] are considered here with a view to extend the usual circuit concept of power.

2. ELECTROMAGNETIC FIELD PROBLEM

In general, an electromagnetic system consists of a structure of bodies in electric and magnetic states having a variety of electric and magnetic properties. Circuits can be regarded as particular electromagnetic systems. Electromagnetic field analysis is required for any electromagnetic system. For non-radiating circuits in particular, the field analysis verifies the assumptions that lead to the description of the electromagnetic interactions in terms of a finite number of variables.

The electromagnetic field in domains with smoothed material properties satisfies Maxwell equations

\[ \nabla \times E = -\frac{\partial B}{\partial t}, \quad \nabla \cdot B = 0, \]
\[ \nabla \times H = J + \frac{\partial D}{\partial t}, \quad \nabla \cdot D = \rho. \]  

(1)

where \( E, D, H, \) and \( B \) have the usual meaning, \( \rho \) and \( J \) are the volume density of the free electric charge and the electric conduction current, respectively.

In order to express several equations in a simpler form, the time derivative of the electric and magnetic flux densities will also be written as:

\[ \frac{\partial D}{\partial t} = \dot{D}, \quad \frac{\partial B}{\partial t} = \dot{B}. \]

(2)

Recall that the electrical displacement current density is given by

\[ J_d = \frac{\partial D}{\partial t} = \dot{D}. \]

(3)

Thus, to the total electric current – conduction plus displacement – corresponds the vector sum of their respective densities:

\[ J_{\text{total}} = J + J_d. \]

(4)

It is important for the extended circuit concept of power to recall that, since

\[ \nabla \cdot (\nabla \times H) = 0, \]

it follows that

\[ \oint_{\Sigma} (J + J_d) \cdot nda = 0. \]  

(6)
Here $\Sigma$ is an arbitrary closed surface that surrounds completely the electromagnetic system analysed. It coincides with the boundary $\partial \Omega$ of the electromagnetic field boundary value problem. Thus, only the sum of the total current intensities, associated to the closed surface $\Sigma=\partial \Omega$, satisfy a completeness relationship (zero sum). The conduction current components verify the law of charge conservation

$$\int \int \int \int \int \int _{\Omega} \rho \, dV = \int \int \int _{\Omega} \left( J \cdot n \right) \, da = - \frac{d}{dt} \int \int \int _{\Omega} \left( D \cdot n \right) \, da$$ \hspace{1cm} (7)

For an $n$-terminal network we characterize the exchange of electromagnetic energy between the network in question and the surrounding system by the power exchanged through the terminals. If the current flowing out of the $k$-terminal has the current intensity $i_k(t)$ and the terminal potential is $\phi_k(t)$, measured with respect to an arbitrary potential reference, we associate the instantaneous electromagnetic power $p_k(t) = \phi_k(t) i_k(t)$ to this terminal. Then the total power exchanged through the terminals, from the $n$-terminal network to the surrounding system, is evaluated as the algebraic sum of all the $n$ partial instantaneous powers, $p_k(t), k=1,2,\ldots,n$. Thus

$$P_{\text{circuit}}(t) = \sum_{k=1}^{n} p_k(t)$$ \hspace{1cm} (8)

It is well known that the electromagnetic interaction between a circuit and its surroundings is performed through the agency of the electromagnetic field. Accordingly, equation (8) that embeds the circuit concept of power, as it is usually mentioned in electromagnetic field theory textbooks [2-3], must be derived from the field expression of power.

If we choose the unit normal outwardly directed, the electromagnetic power is defined as the flux of the Poynting vector out of the closed surface $\Sigma$ that surrounds completely the electromagnetic system in question. Thus

$$p_k(t) = \int \int _{\Sigma} \left( E \times H \right) \cdot n \, dA$$ \hspace{1cm} (9)

Since (8) is associated to the circuit concept of power, equation (9) is said to represent the field concept of power. In fact it represents the general concept of the electromagnetic power, valid whatever system is analysed.

The general power expression (9) reduces, under severe restrictive conditions, to the particular power expression (8). This expression is particular since it is valid only for $n$-terminal networks without radiation.

In some textbooks this reduction is referred to as the reconciliation between the two concepts of power [2-3]. Since the restrictive conditions are in fact boundary conditions satisfied by the electromagnetic field of the $n$-terminal network on the bounding surface $\Sigma$, we will refer to them as circuit boundary conditions.
R. Răduleț was among the firsts if not the first to attack the problem [6]. He treated the one-port network and obtained the conditions under which the flux of the Poynting vector reduces to the product of the voltage by the current intensity. The mathematical model of the theory of transient parameters [4] for circuits with additional losses by eddy currents, skin, proximity and charge relaxation effects, required the proof of the uniqueness of the solutions of the integral equations with transient parameters as kernels (generalized Kirchhoff laws). Accordingly, necessary and sufficient conditions under which (9) reduces to (8) were systematically derived and proved.

2. EXTENDED CIRCUIT CONCEPT OF POWER

2.1 RADIATING CIRCUIT PROBLEM

Consider an $n$-terminal network enclosed by the bounding surface $\Sigma$ shown in Figure 1a. Radiation may take place through part of the bounding surface.

![Partition of the closed surface $\Sigma$ surrounding completely the $n$-terminal network into a “circuit”, where the wires protrude, and a “radiation” part, respectively; b) Partition of $S_{\text{circuit}}$ into a “current”, union of the wire cross section areas, and a “current-free” part, respectively.](image)

Fig. 1 a) Partition of the closed surface $\Sigma$ surrounding completely the $n$-terminal network into a “circuit”, where the wires protrude, and a “radiation” part, respectively; b) Partition of $S_{\text{circuit}}$ into a “current”, union of the wire cross section areas, and a “current-free” part, respectively.

2.2 BOUNDARY PARTITION

We assume that there exist a partition of $\Sigma$ such that all interconnections between the terminals and the surrounding system, protrude only a part of the surface $\Sigma$. $S_{\text{circuit}}$ denotes this part. $S_{\text{radiation}}$ is the complementary part. The bounding closed curve that separates the two surfaces is $\Gamma$. Thus

$$S_{\text{circuit}} \cup S_{\text{radiation}} = \Sigma, \quad S_{\text{circuit}} \cap S_{\text{radiation}} = \emptyset, \quad \bar{S}_{\text{circuit}} \cap \bar{S}_{\text{radiation}} = \Gamma$$

(10)
A further partition of the “circuit” surface into a “current” and a “current-free” surfaces is shown in Figure 1b. The cross section areas of the interconnections carrying the conduction currents are shown enlarged. The areas are denoted by $S_{ck}, k=1,2,…,n$. The union of these disjoint surfaces is $S_{current}$, where the “circuit” surface Thus, the multiple connected “circuit” surface, $S_{circuits}$ is the union of the disjoint surfaces $S_{current}$- and $S_{current-free}$. Therefore

$$S_{circuits} = S_{current} \cup S_{current-free}, \quad S_{current} \cap S_{current-free} = \emptyset,$$

(11)

$$S_{circuits} \cap S_{current} = \bigcup_{k=1}^{n} S_{ck}.$$

### 2.2 CIRCUIT BOUNDARY CONDITIONS

The circuit boundary conditions are those restrictive conditions required by the above-mentioned reconciliation. In the extend concept of power they are verified on $S_{circuits}$ only.

The first circuit boundary condition requires that the voltage is path independent on $S_{circuits}$. This condition can be expressed as:

$$n \cdot (\nabla \times E) = 0 \text{ on } S_{circuits},$$

or equivalently:

$$n \cdot (\partial B / \partial t) = 0 \Rightarrow n \cdot \vec{B} = 0 \text{ on } S_{circuits}. \quad (12a)$$

This condition implies that there are no inductive couplings between the network and its surroundings.

It also implies that the tangential component of the $E$-field can be written as the negative gradient of an electric potential defined on $S_{circuits}$, with respect to an arbitrary reference of potential. Thus, the contribution to the electromagnetic power $p_E$ of the “circuit” part of the surface can be expressed as:

$$p_{circuits}(t) = \iint_{S_{circuits}} (E \times H) \cdot n dA = \iint_{S_{circuits}} ((n \cdot E)n - \text{grad } \phi \times H) \cdot n dA$$

(13)

Since the normal component of $E$-field does not contribute to the surface integral, $\text{curl}(\phi \vec{H}) = (\text{grad } \phi) \times \vec{H} + \phi \text{curl } H$, $\text{curl } H = J + \partial D / \partial t$, from Maxwell equations (1) and the surface integral involving $\text{curl}(\phi \vec{H})$ transforms into a line integral, the “circuit” contribution (13) becomes

$$p_{circuits}(t) = \oiint_{S_{circuits}} \phi \left[ J + \frac{\partial D}{\partial t} \right] \cdot n dA - \oint_{\Gamma} \phi \vec{H} \cdot d\ell$$

(14)
The second circuit boundary condition requires that there are no capacitive couplings between the network and its surroundings through $S_{\text{circuit}}$. This condition can be expressed as:

$$n \cdot \left(\frac{\partial D}{\partial t}\right) = 0 \quad \Rightarrow \quad n \cdot \dot{D} = 0 \text{ on } S_{\text{circuit}}. \quad (15)$$

The third condition confirms that there are no conduction currents through the “current-free” part of $S_{\text{circuit}}$:

$$n \cdot J = 0 \text{ on } S_{\text{current-free}}. \quad (16)$$

The fourth condition is:

$$E_s = 0 \text{ on } S_{ck}, \quad k = 1, 2, \ldots, n. \quad (17)$$

It transforms every $S_{ck}$ into an equipotential surface having the potential $\phi(t)$.

Based on these four circuit boundary conditions, (12), (15)-(17) (8), the “circuit” power component (14) becomes

$$p_{\text{circuit}}(t) = \sum_{k=1}^{n} \phi_k i_k \cdot \oint_{\Gamma} \phi \mathbf{H} \cdot d\ell. \quad (18)$$

The extended circuit concept of power is thus embedded in the novel expression of the electromagnetic power flowing out of the surface $\Sigma$:

$$p_{\Sigma}(t) = \sum_{k=1}^{n} \phi_k i_k \cdot \oint_{\Gamma} \phi \mathbf{H} \cdot d\ell + \iiint_{S_{\text{outside}}} \left(\mathbf{E} \times \mathbf{H}\right) \cdot ndA. \quad (19)$$

In this expression the currents $i_k$ are the conduction currents.

For non-radiating circuits $S_{\text{circuit}}$ extends such that it coincides with the bounding surface $\Sigma$. Ignoring the second condition we get

$$p_{\Sigma}(t) = \sum_{k=1}^{n} \phi_k i_k^{\text{total}}. \quad (20)$$

where

$$i_k^{\text{total}}(t) = \iiint_{S_a} \left(\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}\right) \cdot ndA, \quad k = 1, 2, \ldots, n. \quad (21)$$

are the total currents comprising both conduction and displacement ones. It is well known that in conductors, the power associated with the flow of displacement current is negligible small. If the closed surface $\Sigma$ goes through the dielectric of a capacitor, the displacement current equals the terminal conduction current of the capacitor. Thus, in practice, the total currents in (20) might be replaced by the conduction currents of the $n$-terminal networks [3]. However, we rather prefer...
Răduleț Timotin Tugulea approach encapsulated in the above mentioned circuit boundary conditions. Returning to our extended circuit concept of power (19) we observe that for circuits allowing radiation there is a fifth circuit boundary condition that follows from (6) and (15), namely,

$$\sum_{k=1}^{n} i_k(t) + \iint_{S_{\text{radiation}}} \frac{\partial D}{\partial t} \cdot n dA = 0. \quad (22)$$

If we extend condition (15) for all the closed surface $\Sigma$ or if

$$\iint_{S_{\text{radiation}}} \frac{\partial D}{\partial t} \cdot n dA = 0 \quad (23)$$

the conduction currents will represent a complete system:

$$\sum_{k=1}^{n} i_k = 0. \quad (24)$$

### 3. DISCUSSION AND CONCLUDING REMARKS

The fact that the electromagnetic power has a unique meaning for closed surfaces only is well known. Since the extended circuit concept of power is based on the partition of the closed surface into different open surfaces, the significance of the power evaluated over open surfaces must further be discussed.

If the circulation of $\vec{\phi} \vec{H}$ vanishes along the border $\Gamma$ of the “circuit” part, $S_{\text{circuit}}$, of the bounding surface $\Sigma$,

$$\oint_{\Gamma} \vec{\phi} \vec{H} \cdot d\ell = 0, \quad (26)$$

then, under a set of restrictive conditions, we may evaluate circuit and radiation components of the electromagnetic power flowing out of $\Sigma$ by confining the power flow out of the respective open surfaces. Recall, however, that (19) is derived for the particular case of an $n$-terminal networks verifying specified circuit boundary conditions.

The condition (26) may be regarded as the gauge condition for the electromagnetic power of $n$-terminal networks with radiation.

The gauge condition is verified in some cases of practical interest.

Let us assume that

$$\vec{E}_i = 0 \quad \text{on} \quad \Gamma. \quad (27)$$

Then the contour is equipotential and the gauge condition becomes
\[ \varphi_0 \oint_{\Gamma} \mathbf{H} \cdot d\ell = 0. \] (28)

For instance, if the power is supplied to the network through a shielded multiconductor cable, the shield is usually equipotential and the currents satisfy a completeness condition, fulfilling thus this gauge condition. Actually, if \( \Gamma \) can be taken through a grounded equipotential part of an enclosure, the gauge condition is verified even if the system of currents is not complete for that particular “circuit” surface. Similarly, if \( \Gamma \) coincides with a magnetic wall, \( H_t = 0 \), (26) is verified again.

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REFERENCES