SCATTERING FROM APERTURES: A GENERAL BOUNDARY INTEGRAL EQUATIONS BASED APPROACH

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In the following we present a new variational direct boundary integral equation approach for solving the scattering and transmission problem for dielectric objects partially coated with a perfect conductive layer (PEC) layer. This development originated as a transmission through aperture in PEC casing problems - and was extended to account for the situation of coated arbitrary dielectric bodies. This represents a new development in the range of aperture problems – covering an area eluded at present by most of the commercially available tools. The main idea is to use the electromagnetic Calderon projector along with transmission conditions for the electromagnetic fields. This leads to a symmetric variational formulation that lends itself to Galerkin discretization by means of divergence-conforming discrete surface currents. Whole ranges of numerical experiments were carried on – and results reported here, confirming the efficacy of the new method

1. INTRODUCTION

Article text A dielectric object (scatterer) of finite extension occupies the region $\Omega_s$ of three-dimensional space. Its surface $\Gamma = \partial \Omega_s$ is supposed to be piecewise smooth and Lipschitz-continuous: $\Omega_s$ is a curvilinear Lipschitz polyhedron in the parlance of [1]. This assumption will hold for all relevant CAD-generated geometries in industrial applications. We can distinguish two parts of the surface: a connected part $\Gamma_0$ coated with a thin metallic “mirror” layer that can be regarded as perfectly conducting, and a non-coated part $\Gamma_a$, the so-called aperture(s), see Fig 1. The latter part is to consist of a few connected components, whose closures in $\Gamma$ are disjoint. Moreover the common boundary of $\Gamma_0$ and $\Gamma_a$ is assumed to be a union of curvilinear Lipschitz polygons. The object is composed of a linear, homogeneous, isotropic material with dielectric

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constant $\varepsilon_s$ and permeability $\mu_0$. Outside, in the “air region” $\Omega' = \mathbb{R}^3 \setminus \overline{\Omega}$, we assume the electric properties of empty space. The scatterer is illuminated by a time harmonic plane wave of angular frequency $\omega > 0$. Since all fields will exhibit the same harmonic dependence on time, the scattering problem can be modeled in the frequency domain. Hence, the unknown quantities will be complex amplitudes (phasors). Those of the exciting electric and magnetic field read:

$$\mathbf{E}_i(x) = p \exp(i \mathbf{k} \cdot \mathbf{x}), \quad \mathbf{H}_i(x) = \frac{1}{\omega \mu} \mathbf{k} \times p \exp(i \mathbf{k} \cdot \mathbf{x}).$$

Fig. 1 Cross-section of partly coated dielectric object

Here $\mathbf{k} \in \mathbb{R}^3$ determines the propagation direction and $p$ is the polarization of this incident wave [2].

What we have described above is an electromagnetic compatibility problem, if the PEC coating is viewed as a shielding layer pierced at the aperture(s). We are interested to what extent the incident wave will penetrate through $\Gamma_a$ and trigger electromagnetic fields inside $\Omega_s$. Quantitative information about their strength at points in $\Omega_s$ has to be provided by numerical simulation.

A metal container filled with a fluid provides a typical arrangement that fits the above abstract setting. Some parts of the container’s wall have been removed and replaced by glass or plastics “windows” that do not interfere with the propagation of electromagnetic waves.

In the setting outlined above it is natural to employ a boundary integral equation method, which transforms the field equations in space to integral equations on $\Gamma$. This approach can easily accommodate the unbounded exterior air region and relieves us from meshing $\Omega$, and (parts of) $\Omega'$. These advantages account for the huge popularity of boundary integral equation methods for the simulation of electromagnetic scattering in the frequency domain [3], [2].
Boundary integral equation methods come in many different flavors: direct and indirect formulations and their discretization based on the Nyström technique, collocation or a Galerkin approach. We are going to focus on Galerkin boundary element discretization of a direct boundary integral equation. The main reasons are:

- that the direct method features tangential components of electromagnetic fields as primary unknowns, the very same quantities that occur in the transmission conditions across the aperture.

- that the structure of the resulting discretized equation perfectly matches the inherent symmetry of the coupled scattering problem. This paves the way for theoretical analysis.

We are not the first to tackle the aperture problem outlined above numerically (see [4] and the references cited therein). Approaches based on expansion into spherical harmonics are presented in [5], [6]. However, this only works for very special geometries. More flexibility is offered by the scheme proposed in [4], which is based on the equivalence principle [7]. Yet, this method is of little practical value, because it entails inverting a large dense matrix.

In this paper we outline an approach that is based the Poincare-Stekhlov operators associated with Maxwell's equations in free space. These operators are also known as the electric-to-magnetic mappings. They will be expressed through boundary integral operators and give rise to a coupled variational problem featuring traces of the electric and magnetic field on \( \Gamma \) as unknowns.

Our focus will be on both the derivation of the coupled variational problem and its Galerkin discretization and the performance of the resulting scheme in numerical experiments. We will sketch the theoretical justification for the validity of the coupled problem, but details will be skipped. A comprehensive exposure of the theoretical techniques is given in [8], [9].

2. MATHEMATICAL MODEL

In the time-harmonic setting the behavior of the complex amplitudes of the electromagnetic fields in both \( \Omega \) and \( \Omega' \) is governed by the homogeneous Maxwell equations. Across the aperture \( \Gamma_a \) the usual continuity of tangential components of electric and magnetic field have to be enforced, whereas the tangential component of the electric field phasor vanishes on \( \Gamma_0 \). The model is summed up in the transmission problem, [10]:

\[
\text{curl } e = -i\omega \mu h \quad \text{curl } h = -i\omega \varepsilon e \quad \text{in } \Omega \cup \Omega',
\]

\[
\gamma_e e = 0, \quad \gamma_e h = 0 \quad \text{on } \Gamma_0
\]

\[
\gamma_i e - \gamma_i e = -\gamma_i e, \quad \gamma_i h - \gamma_i h = -\gamma_i h \quad \text{on } \Gamma_a
\]
Here \( \epsilon = \epsilon_0 \), \( \mu = \mu_0 \) in \( \Omega' \), and \( \epsilon = \epsilon_s \), \( \mu = \mu_s \) in \( \Omega_s \). Moreover, we write \( \mathbf{n} \) for the unit normal vectorfield pointing from \( \Omega_s \) into \( \Omega' \), and \( \gamma' \mathbf{u} \) for the tangential trace \( \mathbf{u} \times \mathbf{n} \) of a vectorfield \( \mathbf{u} \) on \( \Gamma \). Note that \( \mathbf{e} \) and \( \mathbf{h} \) stand for the scattered fields in \( \Omega' \): the total fields are obtained by adding the incident wave fields \( \mathbf{e}_i \) and \( \mathbf{h}_i \), respectively. The scattered fields have to satisfy the Silver-Müller radiation conditions (4). The transmission problem (2) – (5) has a unique solution \([11],[12]\).

Introducing the wave numbers (with physical unit \([ \text{m}^{-1}]\))

\[
k_+ = \omega \sqrt{\epsilon_0 \mu_s} \quad , \quad k_- = \omega \sqrt{\epsilon_s \mu_0}
\]

and eliminating the magnetic fields altogether, we end up with the electric wave equations

\[
\text{curl} \mathbf{e} - k_+^2 \mathbf{e} = 0 \quad \text{in} \quad \Omega' / \Omega_s \, , \text{respectively}
\]

Due to the elimination of \( \mathbf{h} \) we have to resort to the magnetic trace operator, \( \gamma'^+ \mathbf{u} := k_+^{-1} \gamma^+ \mathbf{u} \), which is related to tangential traces of the magnetic field. Traces of Maxwell solution will be given a special name: two tangential vectorfields \( \xi, \lambda \) on \( \Gamma \) are called (interior/exterior) Maxwell-Cauchy data, if

\[
\xi = \gamma'^+ \mathbf{u}, \lambda = \gamma^+_s \mathbf{u} \, , \text{where} \ \mathbf{u} \ \text{solves} \ (7) \ \text{in} \ \Omega' / \Omega_s, \ \text{respectively}.
\]

In terms of wave numbers, electric field and magnetic traces, the transmission conditions at \( \Gamma_a \) become:

\[
\gamma'^+ \mathbf{e} - \gamma' \mathbf{e} = -\gamma'^+ \mathbf{e}_i \, , \quad \frac{k_+}{\mu_0} \gamma'^+ \mathbf{e} - \frac{k_-}{\mu} \gamma'^- \mathbf{e} = -\gamma' \mathbf{h}_i \quad \text{on} \ \Gamma_a
\]

A crucial tool will be the Maxwell Poincaré-Steklov operators \( \mathbf{T} \) and \( \mathbf{T}^\prime \), also known as electric-to-magnetic mappings, that take tangential components of the electric field on \( \Gamma_a \) to the magnetic traces of the associated Maxwell solutions in \( \Omega_s \) and \( \Omega' \), respectively. In order to define them properly, we have to establish a suitable framework of function spaces. For a more detailed discussion we refer to \([8]\) and the references cited therein.

The natural energy spaces for the electric wave equations (7) are:

\[
H(\text{curl}, \Omega_s) := \left\{ \mathbf{u} \in \left( L^2 (\Omega_s) \right)^3 \vline \nabla \times \mathbf{u} \in \left( L^2 (\Omega_s) \right)^3 \right\}
\]

\[
H_{\text{loc}}(\text{curl}, \Omega_s) := \left\{ \mathbf{u} \in \left( L^2_{\text{loc}} (\Omega_s) \right)^3 \vline \nabla \times \mathbf{u} \in \left( L^2_{\text{loc}} (\Omega_s) \right)^3 \right\}
\]
where \( \mathbf{H(\text{curl}, \Omega, \gamma)} \) becomes a Hilbert space when equipped with the natural graph norm \( \| \mathbf{H(\text{curl}, \Omega, \gamma)} \| \).

Furthermore one can show that [13]:

\[
H_{\Gamma_0} (\text{curl}, \Omega, \gamma) := \left\{ u \in \mathbf{H(\text{curl}, \Omega, \gamma)} \mid \mathbf{\gamma^\gamma u|_{J_\gamma} = 0} \right\}
\]

defines a closed subspace of \( \mathbf{H(\text{curl}, \Omega, \gamma)} \).

The characterization of the range of \(\gamma^-\) and \(\gamma^+\), in other words, the issue of trace spaces for \( \mathbf{H(\text{curl}, \Omega, \gamma)} \), turns out to be a mathematical challenge. Only recently, a comprehensive answer even for non-smooth domains was given in [14], [15], [16]. We summarize the results as follows: there is a Hilbert space \( \mathbf{H_x^{1/2}(\text{div}_{\Gamma}, \Gamma)} \) of “tangential vectorfields” on \( \Gamma \) such that \(\gamma^- : C^m(\Omega, \gamma) \mapsto TL^2(\Gamma)\) and \(\gamma^+ : C^m(\Omega \cap B) \mapsto TL^2(\Gamma)\) can be extended to continuous and surjective mappings

\[
\gamma^- : H_{\Gamma_0} (\text{curl}, \Omega, \gamma) \mapsto H_x^{1/2}(\text{div}_{\Gamma}, \Gamma) \quad \text{and} \quad \gamma^+ : H_{\Gamma_0} (\text{curl}, \Omega \cap B) \mapsto H_x^{1/2}(\text{div}_{\Gamma}, \Gamma)
\]

Crucial is the self-duality of \( \mathbf{H_x^{1/2}(\text{div}_{\Gamma}, \Gamma)} \): based on the bilinear anti-symmetric pairing

\[
\langle \mathbf{u, v} \rangle_{\Gamma, \Gamma} = \int_{\Gamma} ((\mathbf{u} \times \mathbf{n}) \cdot \mathbf{v}) dS, \quad \mathbf{u, v} \in L^2(\Gamma)
\]

Since the aperture \( \Gamma_a \) is a special part of the boundary, we write \( \mathbf{r_a} \) for the restriction operator \( TL^2(\Gamma) \mapsto TL^2(\Gamma_a) \) and following [15] we define:

\[
\mathbf{H}^{1/2}(\text{div}_{\Gamma}, \Gamma) = \left\{ \mathbf{u} \in \mathbf{TH}^{1/2}(\Gamma) : \text{div}_{\Gamma} \mathbf{u} \in H^{1/2}(\Gamma) \right\}
\]

\[
\mathbf{H}^{1/2}_{x,00}(\text{div}_{\Gamma}, \Gamma_a) := \mathbf{r_a} \left( \mathbf{H}^{1/2}(\text{div}_{\Gamma}, \Gamma) \right)
\]

\[
\mathbf{H}^{1/2}_x(\text{div}_{\Gamma}, \Gamma_a) \equiv \left\{ \phi \in \mathbf{H}^{1/2}_{x,00}(\text{div}_{\Gamma}, \Gamma_a), \mathbf{\tilde{\phi}} \in \mathbf{H}^{1/2}(\text{div}_{\Gamma}, \Gamma) \right\}
\]

where \( \mathbf{\tilde{\phi}} \) is the extension by zero of \( \phi \) on \( \Gamma \). It turns out that \( \mathbf{H}^{1/2}(\text{div}_{\Gamma}, \Gamma_a) \) is a trace space [15].

As announced above, we will now introduce the electric-to-magnetic mappings
\[ T^- : \begin{cases} H^{-1/2}_x (\text{div}_\Gamma, \Gamma_a) \mapsto H^{-1/2}_x (\text{div}_\Gamma, \Gamma) \\
\zeta \mapsto \gamma_N^e \end{cases} \tag{10} \]

where

\[ \text{curl} \text{curl} e - k_e^2 e = 0 \text{ in } \Omega, \quad \gamma_e^- e = \tilde{\zeta} \quad \text{on } \Gamma \tag{11} \]

\( \tilde{\zeta} \) being the trivial extension (i.e. by zero) of \( \zeta \) to \( \Gamma \), and

\[ T^+ : \begin{cases} H^{-1/2}_x (\text{div}_\Gamma, \Gamma_a) \mapsto H^{-1/2}_x (\text{div}_\Gamma, \Gamma) \\
\zeta \mapsto \gamma_N^e \end{cases} \tag{12} \]

where \( e \) satisfies the Silver-Müller radiation conditions (5) and:

\[ \text{curl} \text{curl} e - k_e^2 e = 0 \text{ in } \Omega, \quad \gamma_e^- e = \zeta \quad \text{on } \Gamma \tag{13} \]

It is important to note that \( T \) does not necessarily make sense: If \( k_e^2 \) coincides with a Dirichlet eigenvalue of the differential operator \( \text{curl} \text{curl} \), then the boundary value problem (11) will not have a unique solution. So, whenever using \( T \) we will tacitly make the assumption that the wave number \( k_e^2 \) does not coincide with the square root of an interior Dirichlet eigenvalue of \( \text{curl} \text{curl} \) in \( \Omega \).

Using the transmission conditions (8) along with the definition of the electric-to-magnetic mapping, and setting \( \zeta := \gamma_e^- e \in H^{-3/2}_x (\text{div}_\Gamma, \Gamma_a) \), gives us the equation

\[ r_a \left( \frac{k_e}{\mu_e} T^- \zeta - \frac{k_e}{\mu_0} T^+ \left( \gamma_e^- e \right) - \gamma_e^- h \right) = 0 \text{ in } H^{-3/2}_x (\text{div}_\Gamma, \Gamma_a) \tag{14} \]

We have emphasized that this equation is posed in \( H^{-3/2}_x (\text{div}_\Gamma, \Gamma_a) \) to elucidate that the dual space \( H^{-1/2}_x (\text{div}_\Gamma, \Gamma_a) \) provides the right test functions for a variational formulation. This finally reads: seek \( \zeta := \gamma_e^- e \in H^{-1/2}_x (\text{div}_\Gamma, \Gamma_a) \) such that:

\[ \left< \frac{k_e}{\mu_e} T^- \zeta - \frac{k_e}{\mu_0} T^+ \left( \gamma_e^- e \right), \mu \right>_{\Gamma, \Gamma_c} = \left< \gamma_e^- h, \mu \right>_{\Gamma, \Gamma_c} \quad (\forall) \quad \mu \in H^{-1/2}_x (\text{div}_\Gamma, \Gamma_a) \tag{15} \]
3. ELECTROMAGNETIC CALDERON PROJECTOR

The starting point of the derivation of boundary integral equations is representation formulas involving potentials, that is, mappings of functions on $\Gamma$ to functions on $\Omega' \cup \Omega_\Sigma$. Well known are the scalar and vectorial single layer potentials, whose integral representation is given by $x \notin \Gamma$:

$$
\Psi^h_{V}(\phi)(x) := \int_{\Gamma} \phi(y) G_k(x-y) dS(y)
$$

$$
\Psi^h_{V}(u)(x) := \int_{\Gamma} u(y) G_k(x-y) dS(y)
$$

with the Helmholtz kernel

$$
G_k(x-y) = \frac{e^{j k |x-y|}}{4\pi |x-y|}
$$

It is shown in [8] and [2] that, if $e \in H_{loc}(\text{curl}, \Omega_\Sigma \cup \Omega')$ satisfies:

$$
\text{curl} \text{curl} e - k^2 e = 0 \quad \text{in} \quad \Omega_\Sigma \cup \Omega'
$$

and the Silver-Müller radiation conditions, then the field $e$ can be represented by the so-called Stratton-Chu formula:

$$
e = -\Psi^h_{DL}\left(\begin{bmatrix} \gamma^+ e \\ \gamma^- e \end{bmatrix}_{\Gamma}\right)(x) - \Psi^h_{SL}\left(\begin{bmatrix} \gamma^+ e \\ \gamma^- e \end{bmatrix}_{\Gamma}\right)(x) \quad x \in \Omega_\Sigma \cup \Omega'
$$

using the jump operator $[\cdot]_{\Gamma}$ defined by $[\cdot]_{\Gamma} := \gamma^+ - \gamma^-$ for some trace $\gamma$ onto $\Gamma$ and where we have introduced the (electric) Maxwell single layer potential according to

$$
\Psi^h_{SL}(u)(x) := k \Psi^h_{V}(u)(x) + \frac{1}{k} \text{grad}_x \Psi^h_{V}(\text{div}_x u)(x) \quad x \notin \Gamma,
$$

and the (electric) Maxwell double layer potential

$$
\Psi^h_{DL}(u)(x) := \text{curl}_x \Psi^h_{V}(u)(x), \quad x \notin \Gamma.
$$

Both Maxwell potentials provide radiating solutions of (16). They also allow the application of electric and magnetic trace operators from both sides of $\Gamma$ [8]. This paves the way for defining the boundary integral operators

$$
S_{\gamma} := \{ \gamma \Psi^h_{SL} \}_{\Gamma} \quad C_{\gamma} := \{ \gamma \Psi^h_{DL} \}_{\Gamma},
$$
where \( \{ i \}_\Gamma \) is the average \( \{ i \}_\Gamma := \frac{1}{2}(i^- - i^+) \) for some trace \( i \) onto \( \Gamma \). The operators \( S_k \) and \( C_k \) furnish continuous mappings \( S_k, C_k : H^{3/2}_s(div, \Gamma) \mapsto H^{1/2}_s(div, \Gamma) \) (see [8]).

From (17) it is clear that not all traces can be continuous across \( \Gamma \). More precise information is provided by the jump relations [8].

Now, let us apply the exterior and interior trace operators to the representation formula (17) and use the jump relations. This gives:

\[
\begin{align*}
\gamma_t^- e &= \frac{1}{2} \gamma_t^- e + C_{\kappa}(\gamma_t^- e) + S_{\kappa}(\gamma_N^- e), \\
\gamma_t^+ e &= \frac{1}{2} \gamma_t^+ e - C_{\kappa}(\gamma_t^+ u) - S_{\kappa}(\gamma_N^+ e), \\
\gamma_N^- e &= S_{\kappa}(\gamma_t^- e) + \frac{1}{2} \gamma_N^- e + C_{\kappa}(\gamma_N^- e), \\
\gamma_N^+ e &= -S_{\kappa}(\gamma_t^+ e) + \frac{1}{2} \gamma_N^+ e - C_{\kappa}(\gamma_N^+ e).
\end{align*}
\]

A concise way to write these formulae relies on the Calderon projectors, c.f. [9], [17], and [10],

\[
P_k^- := \begin{pmatrix}
\frac{1}{2} Id + C^{-} & S^{-}
\end{pmatrix}, \quad P_k^+ := \begin{pmatrix}
\frac{1}{2} Id - C^{+} & -S^{-}
\end{pmatrix}
\]

By construction, the operators \( P_k^+ , P_k^- : H^{3/2}_{s}(div, \Gamma)^2 \mapsto H^{1/2}_{s}(div, \Gamma)^2 \) are projectors. Also note that \( P_k^+ + P_k^- = Id \) and that the range of \( P_k^+ \) coincides with the kernel of \( P_k^- \) and vice versa. It was already shown [18] that a pair of functions \( (\zeta, \mu) : H^{1/2}_{s}(div, \Gamma)^2 \times H^{1/2}_{s}(div, \Gamma)^2 \) are interior or exterior Maxwell Cauchy data (i.e a radiating solution of (16)), if and only if they lie in the kernel of \( P_k^+ \) or \( P_k^- \), respectively.

4. COUPLED BOUNDARY INTEGRAL EQUATIONS

Next, we aim to find expressions for the Poincare-Stekhlov operators using the boundary integral operators \( S_k \) and \( C_k \) introduced in the previous section. First, we introduce the scaled traces:
\[(\gamma^+, \lambda^+) = \left( \gamma^+ e, \frac{k^+}{\mu_0} \gamma^+_N e \right), \quad (\gamma^-, \lambda^-) = \left( \gamma^- e, \frac{k^-}{\mu} \gamma^-_N e \right)\]

With this notation the transmission conditions on \( \Gamma_a \) read

\[\zeta^- = \zeta^+ + \gamma^+_t e, \quad \lambda^- = \lambda^+ + \gamma^+_t h,\]

that are set in \( H^{\Delta/2}_{x} (\text{div}, \Gamma_a) \) and respectively \( H^{\Delta/2}_s (\text{div}, \Gamma_a) \).

From (20) we conclude that:

\[
\begin{bmatrix}
-1/2 Id - C_{k^+} & -\frac{\mu_0}{k_+} S_{k^+} \\
-\frac{k_+}{\mu_0} S_{k^+} & -1/2 Id - C_{k^+}
\end{bmatrix}
\begin{bmatrix}
\xi^+ \\
\lambda^+
\end{bmatrix} = 0 \tag{21}
\]

\[
\begin{bmatrix}
-1/2 Id + C_{k^-} & \frac{\mu_s}{k_-} S_{k^-} \\
\frac{k_-}{\mu_s} S_{k^-} & -1/2 Id + C_{k^-}
\end{bmatrix}
\begin{bmatrix}
\xi^- \\
\lambda^-
\end{bmatrix} = 0 \tag{22}
\]

We are looking for a symmetric formulation. Hence, starting from bottom equations (21) and (22), and using to eliminate the magnetic traces remaining on the right hand side afterwards the first equations from (21) - (22) one ends up with:

\[
\lambda^+ = \left[ -\frac{k^+}{\mu_0} S_{k^+} + \left( \frac{1}{2} Id - C_{k^+} \right) \left( \frac{\mu_0}{k^+} S_{k^+} \right)^{-1} \left( -\frac{1}{2} Id - C_{k^+} \right) \right] \xi^+ \tag{23}
\]

\[
\lambda^- = \left[ \frac{k^-}{\mu^-} S_{k^-} - \left( \frac{1}{2} Id + C_{k^-} \right) \left( \frac{\mu^-}{k^-} S_{k^-} \right)^{-1} \left( -\frac{1}{2} Id + C_{k^-} \right) \right] \xi^- \tag{24}
\]

Summing up, we have the representations

\[
\begin{align*}
T^- &= \left[ \frac{k^-}{\mu^-} S_{k^-} - \left( \frac{1}{2} Id + C_{k^-} \right) \left( \frac{\mu^-}{k^-} S_{k^-} \right)^{-1} \left( -\frac{1}{2} Id + C_{k^-} \right) \right] \\
T^+ &= \left[ -\frac{k^+}{\mu_0} S_{k^+} + \left( \frac{1}{2} Id - C_{k^+} \right) \left( \frac{\mu_0}{k^+} S_{k^+} \right)^{-1} \left( -\frac{1}{2} Id - C_{k^+} \right) \right]
\end{align*}
\tag{25-26}
\]

As expected, these operators map continuously
After some computations (detailed in [13]) one can write:

\[
T^\ast \xi = \frac{k^-}{\mu} S_k \xi + \left( \frac{1}{2} \text{Id} + C_k \right) \lambda^-
\]  

(27)

Also from first equation in (22) one can see that we obtain:

\[
\left( -\frac{1}{2} \text{Id} + C_k \right) \xi + \frac{\mu^-}{k^-} S_k \lambda^- = 0
\]

(28)

This equation is set in \( H_{x}^{\frac{3}{2}}(\text{div}, \Gamma) \). It also represents a magnetic to electric mapping.

We also get:

\[
T^\ast \left( \xi - \gamma^e_i e_i \right) = -\frac{k^+}{\mu_0} S_k \left( \xi - \gamma^e_i e_i \right) + \left( \frac{1}{2} \text{Id} - C_k \right) \lambda^+
\]

(29)

and from first equation in (21):

\[
\left( \frac{1}{2} \text{Id} + C_k \right) \left( \xi - \gamma^e_i e_i \right) + \frac{\mu_0}{k^+} S_k \lambda^+ = 0
\]

(30)

Also this equation is set in \( H_{x}^{\frac{3}{2}}(\text{div}, \Gamma) \). As with (27) this equation is also a magnetic to electric mapping.

By plugging (27) and (29) into (14) we get:

\[
\left[ \frac{k^-}{\mu} S_k \xi + \left( \frac{1}{2} \text{Id} + C_k \right) \lambda^- \right] - \left[ -\frac{k^+}{\mu_0} S_k \left( \xi - \gamma^e_i e_i \right) + \left( \frac{1}{2} \text{Id} - C_k \right) \lambda^+ \right] = \gamma^e_i h_i
\]

(31)

Equation (31) is also set in \( H_{x}^{\frac{3}{2}}(\text{div}, \Gamma_0) \).

**Note** Equation (31) – is the transmission condition – effective only on \( \Gamma_0 \).

Equations (28) and (30) involve relations between integral operators defined on the whole of \( \Gamma \).

The equations (28), (30) and (31) are the equations to be solved. We set them as a system of equations as follows:
We will test now the system of equations against functions set as follows: First equation is tested with functions $\mu \in \mathbf{H}^{\frac{1}{2}}(\text{div}, \Gamma_0)$. Second and third equations are tested with functions $\xi, \nu \in \mathbf{H}^{\frac{1}{2}}(\text{div}, \Gamma)$. 

Remark. Taking into account the unique solvability of the transmission problem, the assumptions made on the values of the wave numbers are undesirable. They are related to the phenomenon of “forbidden frequencies” [20] or “spurious resonances” that haunt most variational formulations of scattering transmission problems. A profound analysis of the impact of spurious resonances in the case of electromagnetic scattering is given in [21].

In fact, when facing a spurious resonance, the solutions for $\lambda$ may no longer be unique, but the fields recovered through the representation formula (17) will. Nevertheless, spurious resonances are worrisome, because they involve a loss of stability that will lead to singular or extremely ill-conditioned linear systems after discretization.

Elegant ways to avoid spurious resonances in the case of a purely exterior scattering problem are combined field integral equations [2]. Unfortunately, an analogous stable formulation for the transmission problem has hitherto not been found.

5. GALERKIN BOUNDARY ELEMENT DISCRETIZATION

We aim to use a conforming Galerkin boundary element discretization of (32). To that end, $\Gamma$ will be approximated by a triangulation $\Gamma_h$ composed of flat triangles. We assume that the boundary of $\Gamma_d$ is approximately resolved by edges of $\Gamma_h$. 

\[
\begin{pmatrix}
\frac{k}{\mu} S_{k'} + \frac{k^+}{\mu_0} S_{k'} \\
\frac{1}{2} Id + C_{k'} \\
\frac{1}{2} Id + C_{k'}
\end{pmatrix}
\begin{pmatrix}
\frac{1}{2} Id + C_{k'} \\
\frac{\mu^-}{k^-} S_{k'} \\
0
\end{pmatrix}
\begin{pmatrix}
\frac{1}{2} Id + C_{k'} \\
0 \\
\frac{\mu_0}{k^+} S_{k'}
\end{pmatrix}
\begin{pmatrix}
\xi \\
\lambda^- \\
\lambda^+
\end{pmatrix}
= \begin{pmatrix}
\gamma_h \mathbf{h} + \frac{k^+}{\mu_0} S_{k'} (\gamma^- \mathbf{e}_i) \\
0 \\
\frac{1}{2} Id + C_{k'} (\gamma^+ \mathbf{e}_i)
\end{pmatrix}
\]

(32)
Next, we have to construct a finite dimensional subspaces $\mathbf{v}_h \subset H^{-1/2}_x(\text{div}, \Gamma_h)$ and $\mathbf{vv}_h \subset H^{-1/2}_x(\text{div}, \Gamma)$ that contain piecewise polynomial surface vector fields and possess locally supported basis functions. To motivate their construction, we look at $H(\text{curl}, \Omega)$-conforming finite element schemes for the approximation of electric and magnetic fields. The simplest is provided by the so-called edge elements [22]. Keeping in mind that $H_0^{\text{curl}}(\Omega) = \gamma_0^c \left( \mathbf{H}(\text{curl}, \Omega) \right)$, we simply take the tangential traces of edge element functions on a mesh $\Omega_h$ with $\Omega_h|_\Gamma = \Gamma_h$ as space $\mathbf{vv}_h$. This will give a space of piecewise linear vectorfields on $\Gamma$ whose “surface normal components” are continuous across edges of triangles. This is a well-known sufficient condition for $\mathbf{vv}_h$. The local shape functions on a triangle $T$ are given by the formula

$$b_{i,j}^T := \lambda_i \text{curl}_T \lambda_j - \lambda_j \text{curl}_T \lambda_i, \quad 1 \leq i < j \leq 3$$

where $\lambda_i, i = 1, 2, 3$, are the local linear barycentric coordinate functions in $T$. These basis functions are sketched in Fig. 2. They are associated with the edges of $\Gamma_h$ so that $\text{dim} \mathbf{vv}_h$ will agree with the total number $N$ of edges of $\Gamma_h$. Note that $\mathbf{vv}_h$ agrees with the lowest order div-conforming Raviart-Thomas elements in 2D, c.f. [23]. In electromagnetic scattering problems $\mathbf{vv}_h$ is known as the space of Rao-Wilton-Glisson (RWG) boundary elements [24].

In order to find $\mathbf{v}_h$ recall that an edge element subspace of $H_0^{\text{curl}}(\text{curl}, \Omega)$ can be obtained by dropping all basis functions associated with edges on $\Gamma_0$. As
\[ \mathbf{H}^{-\frac{1}{2}}(\text{div}, \Gamma_a) = \gamma_t\left( \mathbf{H}_{\Gamma_0}(\text{curl}, \Omega) \right), \]

\( \mathbf{v}_k \) will be spanned by all basis functions (33) belonging to edges in the interior of \( \Gamma_a \). Let \( N_a \) denote their number.

When plugging expressions (32) into the bilinear form and also using the fact that the test functions will be exactly the expansion functions – defined for the corresponding surfaces the bilinear form will turn into a linear system of algebraic equations whose general form is given by:

\[ \mathbf{A} \cdot \mathbf{x} = \mathbf{B} \]

where \( \mathbf{A} \) is the matrix of the system, complex, quadratic, of dimension \( (N_a + 2 \times N) \times (N_a + 2 \times N) \), \( \mathbf{x} \) is the vector of unknowns and \( \mathbf{B} \) is the vector representing right hand side term. These vectors are of dimension \( N_a + 2 \times N \).

Hence the general system of equations, in terms of the dimensions of the blocks composing it (as can be seen when compared with (32)) is given by:

\[ \begin{bmatrix}
[N_a \times N_a] & [N_a \times N] & [N_a \times N] \\
[N \times N_a] & [N \times N] & [N \times N]
\end{bmatrix}
\begin{bmatrix}
[N_a,1] \\
[N,1]
\end{bmatrix}
= 
\begin{bmatrix}
[N_a,1] \\
[N,1]
\end{bmatrix}
\]

Eventually, the transmission problem has been converted into a square linear system for the \( 2N + N_a \) unknown coefficients corresponding to surface currents crossing edges of \( \Gamma_0 \); on \( \Gamma_0 \) each edge bears two unknowns, each interior edge of \( \Gamma_a \) has three of them.

Remark. Surface edge elements enjoy a number of unique stability properties owed to their nature as discrete differential forms, see [8] and [25]. This makes it possible to show the quasi-optimality of Galerkin solutions provided that the mesh \( \Gamma_h \) is fine enough.

6. NUMERICAL RESULTS

The new discrete boundary integral formulation is tested numerically for several different arrangements, namely a metallic rectangular container, partially covered and filled with sea water, and a metallic box with four slots.

Throughout the linear system of equations arising from the boundary element Galerkin discretization was solved iteratively using GMRES for complex matrices [26]. Its termination criterion was a relative drop of the Euclidean norm of the residual by a factor of \( 10^4 \). Neither preconditioning nor acceleration of matrix-
vector products by means of fast multipole techniques has been used, because the focus was on assessing the accuracy of the method.

The whole procedure was implemented in C++ using the Microsoft Developer Studio V6.0 and runs on an Intel Xeon Machine, under Windows 2000 operating system. The system has 2GB of RAM memory. The implementation is based on building first the full matrix and then solving it via an iterative solver. To exemplify, for a mesh yielding 3440 unknowns the total solution time was of 415 s of which 193 for building the matrix, 140 for building the RHS, 41 s for solving the system (employing for this case 396 iterations), the rest being covered by postprocessing tasks.

The first geometry considered was a metallic rectangular container filled with sea water. The upper part is partially covered with a metallic lead as can be seen in figure Fig. 3. We approximated the dielectric constant of the water to have only real part and to it to be $\varepsilon_r = 80$. To the authors best knowledge this configuration can only be analyzed with the present proposed approach. The excitation is a plane wave, linearly polarized and propagating in positive z-direction:

$$\mathbf{e} = \mathbf{e}_0 \exp(-jkr) \quad \mathbf{e}_0 = e_0 \hat{\mathbf{x}} \quad e_0 = 1V / m \quad k = k\hat{z}$$

Fig. 3 Sketch of the geometry for the metallic container filled with sea water.

We started by assessing the stability of the solution with respect to the discretization. Meshes range from very coarse (471 unknowns) to fine (4858 unknowns). Results are presented in Fig. 4 for an incident wave of wave number $k=3$ and in figure Fig. 5 for an incident wave of wave number $k=4.5$. One can see that far from resonance frequencies ($k=3$) the influence of mesh coarseness is negligible - whilst close to a resonance frequency ($k=4.5$) the influence of mesh coarseness becomes significant - in the sense that coarse meshes yield far poorer results.
In the followings we present a plot of the electromagnetic field on the surface of the aperture (Fig. 6) as well as of the transmitted field (Fig. 7) for a wave number \( k = 4.5 \). A frequency sweep was done – and results in the form of electric field on a line passing through the centre of the enclosure are presented for several wave numbers in Fig. 8.

To further assess the results of our approach we considered a also a metallic casing (i.e. a PEC rectangular cavity) with four thin slots, as can be seen in Fig. 9. The electromagnetic shielding effectiveness study provides means to help reduce electromagnetic emission or improve the immunity of components present inside the casings. The electric field shielding factor (EFS) computed can be computed as in (35),

\[
EFS(x) = -20 \log \left| \frac{E}{E_1} \right| \quad [dB]
\]  

For this situation the EFS parameter was computed in one point situated in the centre of the structure.

![Fig. 4 Influence of mesh refinement on accuracy of solution, for a metallic container filled with sea water. Wave number \( k = 3.0 \).](image)
Fig. 5 Influence of mesh refinement on accuracy of solution, for a metallic container filled with sea water. Wave number $k = 4.5$.

Fig. 6 Modulus of tangential electric field on the surface of the aperture [$V/m$]. Wave number $k = 4.5$. 
Scattering from Apertures: General BIE Approach

Fig. 7 Modulus of transmitted electric field [V/m]. Wave number $k = 4.5$.

Fig. 8 Modulus of transmitted electric field [V/m] on a line perpendicular to the aperture, passing by the center of the casing.
Such kind of configurations appear often in electronic circuits - and were the object of several investigations [27], [28]. The former takes advantage of the special configuration of the geometry to perform an eigen mode expansion of the solution (so called Modal Method [MM]). In the case of the above cavity (Fig. 9) the excitation is a plane wave propagating perpendicular to the plane of the slots (situated at $z = 0$ and $z = 30$ cm) and with horizontal polarization. Results of our computation (2330 unknowns) as well as the ones from [27] are plotted in Fig. 10. One can observe a good agreement with the semi-analytic computations.

![Fig. 9 Rectangular cavity with four slots. All dimensions in cm.](image)

![Fig. 10 Shielding efficiency of a metallic box of dimensions (60 x 12 x 30), with four slot openings of (20 x 8) cm placed at $z = 0$ and respectively at $z = 30$, measured in the center of the enclosure (30, 6, 15) cm.](image)
... CONCLUSIONS

In this article we outline a direct boundary integral equations approach for transmission problems that is based on electric-to-magnetic mapping operators. As unknowns we employ the tangential traces onto the surface of the scatterer of electric and magnetic fields. Thus boundary transmission conditions are easily accommodated. The surface fields are linked by the electric to magnetic Dirichlet to Neuman (DtN)-type mappings associated with both the scatterer and the exterior space. This leads to a symmetric variational formulation which lends itself to Galerkin discretization by means of divergence conforming discrete surface currents. A couple of numerical experiments confirm the efficacy of the new method. Preconditioning and acceleration of matrix-vector products constitutes the subject of further developments.

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