A GLIMPSE INTO SOME ASPECTS OF NONLINEAR CIRCUIT ANALYSIS

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Abstract Three topics of teaching nonlinear circuit analysis (nonlinear oscillations, fast finding of the periodic steady state, and harmonic balance analysis of strongly nonlinear circuits) are addressed using simple examples solved with PSPICE, MAPLE and APLAC software packages.

1. Introduction

The wide use of the nonlinear devices for the design of modern Systems in a Package or Systems on a Chip requires a deep understanding of the nonlinear phenomena and their simulation methods. Some new efficient analysis methods using intricate mathematical developments have been implemented in software packages like CADENCE and ADS in the last five years. At the same time teaching of nonlinear circuit analysis must take into account the significant drop in the engineering enrollments which has been observed worldwide in the last decade [1]. This loss of interest in our profession brings less motivated students in the Electrical Engineering Faculties. The course presentation and the seminar work must seem attractively and easily understandable to these students. Moreover, the integration of the Europe’s teaching systems in the Bologna framework, leading to a reduction of the bachelor studies to three or four years, requires shorter ways leading to the understanding and using of nonlinear circuit analysis.

The numerical methods, implemented in some known software packages like SPICE or HARMONICA have their well established place in teaching nonlinear circuit analysis. It is well known that analytical solutions give a better insight into circuit properties than a collection of numbers resulting from a numerical method. But only the solutions of some simple first order circuits can be obtained efficiently by hand computation. Moreover, for the most today students that don’t have strong analytical computation skills, automated analytical computation may be very helpful.

Our Circuit Analysis course [6, 7] has a structure which is similarly to [5]. The majority of topics (as operating points of linear and nonlinear resistive circuits, small signal and large signal behavior of nonlinear circuits, frequency domain analysis of AC circuits, transient analysis of nonlinear circuits, chaotic behavior of non-autonomous circuits) are illustrated using PSPICE simulations only [8, 9].

This paper address three topics of teaching nonlinear circuit analysis: nonlinear oscillations, fast finding of the periodic steady state, and harmonic balance analysis of strongly nonlinear circuits. The PSPICE capabilities are not enough for the study of these topics so that MAPLE and APLAC are added. Section 2 is dedicated to the symbolic and numerical analysis of a series RLC oscillator including a nonlinear resistor. Some methods for the fast finding of the initial conditions corresponding to the periodic steady state are discussed in Section 3. The limitations of the harmonic balance analysis are discussed in Section 4 using a simple circuit whose nonlinearity can be changed between a weak one and a strong one.

2. Series RLC oscillator

This circuit has a passive linear inductor, a passive capacitor, and a locally active nonlinear resistor as, for example, in Fig. 1.

Using the state equations it is proved that his circuit has an unique equilibrium point in the origin which is an unstable focus or an unstable node. This behavior (B1) is valid around the origin, i.e. for small $i_L = i_R$ ($i_L$: inductor current, $i_R$: resistor current). For relatively large $i_R$, the resistor has a positive dynamic resistance and the circuit has a stable behavior (B2) i.e. the trajectories approach the origin.

In order to facilitate the understanding of the nonlinear oscillation, a solved exercise contains the analysis of two linear circuits: one corresponding to B1 and another corresponding to B2. The symbolic expressions of $i_L(t)$ and $u_C(t)$, the numerical eigenvalues and the phase portraits are obtained for $L=1H$, $C=1F$ and $R=-1/10\Omega$, or $R=1/10\Omega$, the origin being an unstable or a stable focus. The numerical analysis of the nonlinear RLC oscillator (the resistor having the piecewise linear characteristic in Fig. 1) is performed. The phase portrait for two initial states (one inside the limit cycle and the other outside it is drawn. The similarity with the qualitative behavior of the linear circuit with $R>0$ and $R<0$ can be easily observed. All these computations are performed with MAPLE, which has been chosen for its outstanding symbolic computation capabilities.
The following exercises must be solved:

**Exercise 1** Choose numerical values for \( u'(0)=R, L \) and \( C \) so that an unstable node is obtained in the origin.

**Exercise 2** Perform the numerical analysis of the Van der Pol oscillator (having a nonlinear resistor with \( u=\frac{i^3}{3} \)) and obtain the phase portrait. Compare with the phase portrait in the Exercise 1. In the case \( \varepsilon = \sqrt{C/L} \to 0 \) use the variable change \( \tau = t/\sqrt{LC} \) and obtain a linear second order equation of \( i(t) \) deleting the negligible terms. Find the analytical solution of this equation and show that its amplitude depends on the initial state, unlike the nonlinear case.

**Exercise 3** Explain the jump phenomenon for the RC relaxation oscillator using the phase portrait of the series RLC circuit with \( L \to 0 \).

### 3. The periodic response of non-autonomous circuits

This assignment deals with the fast computation of the periodic response (PR) of nonlinear circuits. If the greatest time constant \( \tau_{\text{max}} \) of a linear circuit has a value \( \tau_{\text{max}}>T \) (T -excitation period), the numerical integration of the circuit equations until all transients decay (brute force method) requires the sweeping of about \( N=5\tau_{\text{max}}/T \) excitation periods; this amount of CPU time may be too large for an efficient simulation. The same problem is encountered if nonlinear circuits are analyzed. To reduce the computation time in these cases the shooting methods are used to compute the initial state corresponding to the periodic response (ISPR). The periodic response can be found integrating the circuit equations over a single excitation period starting from ISPR. Our examples are worked out with shooting by exponential extrapolation [10]. If the imaginary part \( \omega_0 \) of a (local) natural frequency \( \omega = \alpha + j\omega_0 \) is close to the excitation angular frequency \( \omega_0 \) all shooting methods fail. In these cases we use an ISPR identification method for linear circuits [11]. A more detailed presentation of these topics is given in the Appendix.

The theoretical presentation contains the following items:

- the non-autonomous circuits with non-customary and customary behavior and some outstanding properties of the last ones [3]:
  - the circuit has an unique periodic response of the same period as its excitation, for any initial state,
  - the exponential – transient – decay,
  - the spectrum conservation.
- the response of a linear circuit at sinusoidal excitation, - ISPR, - shooting by exponential extrapolation.

Two solved exercises are presented. The first one requires the determination of ISPR for a linear circuit of the fourth order. This exercise is solved by two methods. For ISPR identification we need the state equations. To this end the circuit equations are written manually. All circuit variables except the state ones are eliminated using MAPLE and the normal form of the state equations is obtained. The analytical solution of these equations is obtained with MAPLE in terms of the ISPR components \( \chi_i(0), i=1,\ldots,4 \). The ISPR is identified solving a linear system with MAPLE [11].

The same problem is solved with shooting by exponential extrapolation. This method requires the estimation of ISPR using the averaged state variables corresponding to three excitation periods. This computation, together with the determination of the steady state error is done using a procedure written in MAPLE programming language. The PSPICE software, called from another MAPLE procedure, is used to perform the circuit analysis. The number of analyzed periods to reach the steady-state with a relative error of \( 10^{-3} \) is compared with the period number for the method of brute force used by PSPICE. It follows that in this case shooting by exponential extrapolation is 30 times faster than the brute force.

The second solved problem requires the determination of ISPR of a first-order nonlinear circuit. The problem is solved by two methods: shooting by exponential extrapolation and the brute force method. It follows that in this case shooting by exponential extrapolation is 27 times faster than the brute force.
E = 1.2\sin(2\pi ft), f=1 Mz, R1=1k \Omega, C1=6.9\mu F.

Fig. 4 First order nonlinear circuit for fast finding of the periodic response

In the first proposed exercise it is required to modify the excitation angular frequency \( \omega \) for the circuit in Fig.3 in order to be close to a value \( \omega_k \) and to point out the inefficiency of the shooting by exponential extrapolation in this case. The second proposed exercise requires the analysis of a diode rectifier with a series inductor filter with the brute force and shooting by exponential extrapolation. The ISPR obtained using the brute force method and the identification method must be compared. As SPECTRE RF is not available in our laboratory, the shooting approach to ISPR computation has been illustrated using MAPLE and PSPICE.

4. The analysis of nonlinear circuits in the frequency domain

The harmonic balance (HB) is the classical frequency domain analysis method for the nonlinear circuits. It is well known that HB works very well if PR has relatively few harmonic components but has convergence problems for waveforms with abrupt transitions (having a relatively large number of harmonic components) and for circuits with strongly nonlinear elements [7]. These facts are illustrated by the solved exercise requiring PR computation for a first order nonlinear circuit (Fig.4). The problem is solved by HB and the brute force time domain analysis. A piecewise-linear characteristic with two segments is considered for the diode. Two input waveforms are tested: pure sinusoidal and AM (\( \sin(2\pi f_1t)\sin(2\pi f_2t) \), \( f_1=1\text{KHz}, f_2=10\text{MHz} \)). The results of the HB simulator from APLAC 7.70 Student Version using the maximum allowable order of the intermodulation products (9) [12] are compared to the PSPICE ones. The linear segment slopes of the diode are changed in order to observe the influence of the weak or strong nonlinearity.

For the pure sinusoidal input there is a good agreement between the two approaches for the direct resistance \( R_d \in (0.1\Omega, 10\Omega) \) and the inverse resistance \( R_i \in (10K\Omega, 10M\Omega) \). For the AM input APLAC is not able to show the high frequency detail of the capacitor voltage. As the ratio \( R_i / R_d \) increases (the nonlinearity becomes stronger) the APLAC envelope becomes less accurate. For example if \( R_d = 1\Omega, R_i = 1M\Omega \) the amplitude of the APLAC envelope is less than 50% of the PSPICE one.

![Fig. 5. PSPICE capacitor voltage for AM input (a-envelope, b-high frequency detail in the up envelope zone) for \( R_d=0.1\Omega, R_i=1M\Omega \)](image)

There are two proposed exercises:

**Exercise 1** Perform the analysis of the circuit in Fig. 4 with AM input having \( f_1=50\text{KHz}, f_2=150\text{KHz} \). Explain the very good agreement between PSPICE and APLAC 7.70 in this case.

**Exercise 2** Analyze the same circuit for \( f_1=1\text{KHz}, f_2=10\text{KHz} \). Explain the approximate agreement between PSPICE and APLAC 7.70 in this case.

The set of frequencies considered by APLAC and the maximum order of the intermodulation products for the student version (9) must be taken into account for these explanations.

**References**

Appendix Fast computation of the periodic response

Steady state error We consider that the steady state is reached if the relative error $\varepsilon$ between the responses in $(m+1)$-th and $m$-th periods does not exceed an imposed limit $\varepsilon_0$, $\varepsilon \leq \varepsilon_0$ where $\varepsilon = \max \varepsilon_j$, $n$ is the number of components in the state vector $x$,

$$
\varepsilon_i = \frac{c_i^{(m+1)} - c_i^{(m)}}{c_i^{(m+1)}} \text{, and } c_i^{(m)} = \sqrt{\int_{mT}^{(m+1)T} x_i^2(\tau) d\tau} .
$$

Shooting ISPR is estimated using the response corresponding some excitation periods. The integration starts with this estimated value (Fig. 6). This procedure is used for each state variable. Shooting with Newton-Raphson (implemented in SPECTRE RF) and shooting by linear extrapolation are the most known. It has been proved [10] that exponential extrapolation is more efficient than the other shooting procedures.

Exponential extrapolation The algorithm of shooting by exponential extrapolation has the following steps:
1 - three excitation periods 1, 2, 3 are swept using a stiff-stable method and the average values $\bar{x}_j(1), \bar{x}_j(2)$, $\bar{x}_j(3)$ corresponding to each period are computed for all state variables ($j=1,...,n$),
2 - the asymptotic values $a_j$ are computed with the relation $a_j = \frac{\bar{x}_j(1)\bar{x}_j(3) - \bar{x}_j^2(2)}{\bar{x}_j(1) + \bar{x}_j(3) - 2\bar{x}_j^2(2)}$, $(j=1,...,n)$

Fig. 6. The shooting approach 3 - ISPR components $x_{j0}^*$ are computed as $x_{j0}^* = x_j(3) + a_j j=1,...,n$ where $x_j(3)$ are the initial conditions in the period 3,
4 - two periods 4 and 5 are swept starting from $x_{j0}^*$, $j=1,...,n$,
5 - if the steady state error between the responses in the periods 4 and 5 does not exceed an imposed limit $\varepsilon_0$ then the periodic response is obtained, if not, the computation continues with the first step.

ISPR identification Consider a linear circuit having an unique periodic response of the same period $T$ as its excitation. It can be shown easily that the average value of a state variable can be written as

$$
\bar{x}_k = x_k(0)A_{k1} + x_k(0)A_{k2} + \ldots + x_k(0)A_{kn} + A_{k,n+1}
$$

for $k=1,2,...,n$ where $\bar{x}_k = \frac{1}{T} \int_0^T x_k(t) dt$, $x_k(0)$ ($k=1,...,n$) are the components of ISPR, and $A_{kj}$ are constant coefficients.

Sweeping $n+1$ excitation periods with a numerical integration method and computing $\bar{x}_j(k=1,...,n)$ for each period, a system of $n(n+1)$ equations is obtained. Solving this system $A_{kj}(i=1,...,n, \ j=1,...,n+1)$ are computed.

The circuit state equations in the normal form are $\dot{x} = Ax + Bu$. Averaging this expression it follows $A\bar{x} + B\bar{u} = 0$. Replacing each $x_k$ with its expression in terms of $x_k(0)$ ($k=1,...,n$) we obtain an equation system whose solving gives ISPR.