Study on the Magnetic Field Produced by Coils of Certain Shapes

Ioana-Gabriela SÎRBU, Lucian MANDACHE
University of Craiova, Electrical Engineering Faculty, Decebal Blv. 107, Craiova; osirbu@elth.ucv.ro

Mihai IORDACHE
University POLITEHNICA Bucharest, Electrical Engineering Faculty, Spl. Independenței 313, Romania

Abstract. This paper proposes to determine analytically or numerically the magnetic field produced by the coil windings. In this context the estimation of the magnetic field strength produced by one single turn of different shapes has an important role. One starts from the magnetic field strength produced by a circular single turn, then the principle of calculation was generalized for the elliptic single turns. This study is useful for the finite length coils’ case.

1 Introduction

The role of a solenoid in an electromagnetic system is to create a magnetic field. Depending on its geometry and its position relative to the other components of the system one can obtain magnetic fields that have certain distributions or specific orientations. Therefore its behaviour and its exact role in the operation of such a system must be known with accuracy.

The fundamental element of a solenoid is the single turn coil. Its behaviour, when it is passed by an electrical current, was studied during the time by many theorists that proposed different solutions to solve this problem. One can mention here the possibility of treating this coil as a sufficient number of conducting segments connected each other those contribution is summing up [1], and the analysis based on a numerical modelling of the magnetic field produced around it [2]. These studies proceed specially from the analysis of some shapes often met in practice, the obtained results being presented in works as [3], [4], [5], [6], [7]. Whatever it was the method of solving this problem, analytically, numerically or experimentally, it remains a problem of actuality. In order to tackle it a good knowledge both of the electromagnetic field theory [8] and of the mathematic calculus instruments (analytic, geometric and numeric) is necessary.

In this overall idea this work proposes to offer an alternative possibility to evaluate the magnetic field produced by a single turn coil. Two different shapes of practical importance are analysed here: the circle and the ellipse. For each of these cases one follows to determine the magnetic field strength $H$ in a point placed on a straight line that passes through the centre of the coil, perpendicular or not. The obtained mathematical formulae were solved analytically or using numerical techniques of calculation. The results were analysed comparatively, inclusively using specialized programs for the electromagnetic field modelling (QuickField), thus proving the correctness and the utility of the proposed formulae.

2 Analysis of the Circular Single Turn Coil

Our study begins with the case of the circular coil (fig.1), easier to be analysed due to its geometric symmetry. One specified firstly the procedure of calculation the magnetic field in the points placed on a straight line (axis) perpendicular on the coil plane. One passes that to a generalisation of the results, for points placed on a straight line inclined with a certain angle with respect to the coil plane that passes through the centre of the circle. Practically we are interested specially by the field oriented along this line, so that our study focuses on this component.
Case of a perpendicular axis

The determination of the magnetic field strength $H$ in a point placed on the coil axis, perpendicular on its plane, is a problem already solved [5]. This magnetic quantity is determined starting from the Biot-Savart-Laplace’s formula [8]:

$$ H = \frac{I}{4\pi} \oint \frac{\vec{d}r \times \vec{R}}{R^3}, $$

where $I$ represents the intensity of the electric current through the coil, and $R$ is the distance from the point $P$ to the elementary section of the coil $d\vec{r}$, oriented in the direction of the current (fig.1, (a)).

Taking into account the symmetries and knowing that the components of the magnetic field on the direction perpendicular on the axis cancel each others, the magnetic field strength in the point $P$ situated at a distance $h$ from the centre of the coil of radius $a$ is oriented following the direction of the axis and it is calculated with the analytic formula [5]

$$ H = \frac{I \cdot a^2}{2 \left( a^2 + h^2 \right)^{3/2}}. $$

The variation curve of $H$ as function of the position of the point $P$ on the axis was represented in fig. 1, (b).

Case of an axis inclined with a given angle

In case of an axis inclined with an angle $\alpha$ with respect to the coil plane (fig. 2) the problem becomes more complex, because some of its symmetry is lost. Thus the magnetic field strength will not have only one component along this line, but also a component on a direction perpendicular on it. Because this second component is less important in the practical cases, its study will not be presented here.
The distance between the point P to two points diametrically opposite situated on the coil (fig. 2) is in this case different. By geometrical reasons the values of these two distances are calculated with the formulae:

\[ R' = \sqrt{a^2 + h^2 + 2ah \cdot \cos \alpha \cdot \cos \theta}, \quad R'' = \sqrt{a^2 + h^2 - 2ah \cdot \cos \alpha \cdot \cos \theta} \]  

(3)

With the Biot-Savart-Laplace’s formula, the magnetic field strength on the direction of the line, produced by two elementary sections diametrically opposite of the coil, \( \vec{d}r' \) and \( \vec{d}r'' \), are calculated with the relations

\[
dH_x' = \frac{I \cdot a \cdot d\theta}{4\pi R'^2} \cdot \sqrt{1 - \left( \frac{h \cdot \cos \alpha \cdot \sin \theta}{R'} \right)^2} \cdot \sqrt{1 - \left( \frac{h + a \cdot \cos \alpha \cdot \cos \theta}{R'} \right)^2},
\]

\[
dH_x'' = \frac{I \cdot a \cdot d\theta}{4\pi R''^2} \cdot \sqrt{1 - \left( \frac{h \cdot \cos \alpha \cdot \sin \theta}{R''} \right)^2} \cdot \sqrt{1 - \left( \frac{h - a \cdot \cos \alpha \cdot \cos \theta}{R''} \right)^2},
\]

(4)

where \( \theta \) is the angle measured in the centre of the coil (fig. 2).

The estimation of the global magnetic field in a point (tangential component) is made afterwards by summing up the contribution of the elementary pairs of sections, on a semicircle:

\[
H_x = 2 \int_0^{\pi/2} dH_x, \quad \text{where} \quad dH_x = dH_x' + dH_x''.
\]

(5)

Figures 3 and 4 represent the results obtained by analysing this situation. For verification, we made also the determination of the magnetic field distribution produced by a circular coil using the software package QuickField Professional (fig. 3). This permits to obtain the variation of \( H \) along any straight line, inclined or perpendicular on the coil.

![Figure 3: Magnetic field strength estimation along any inclined axis, using QuickField Professional.](image)

![Figure 4: Magnetic field strength along an inclined axis: comparative analysis for \( \alpha = \pi/3 \) (a); using formula implemented in MATLAB (b).](image)
Figure 4 (a) presents the comparative analysis of the data obtained in QuickField Professional and using the relations (4)-(5) implemented in MATLAB. Due to the complexity of the integral formula it is very difficult to offer an exact solution of it. For this reason we applied a procedure of numerical integration using the composed formula of the trapeziums [9]. One can remark a good approach of the values obtained using these methods. Figure 4 (b) shows the variation curve of $H$ along the straight line, for different angles of its inclination.

3 Analysis of the Elliptic Single Turn Coil

The ellipse can be shown as a more general case of the circle. Consequently the manner of solving the problem of determination of the magnetic field produced by an elliptic coil passed by a current will be similar to those presented previously, but adapted to the geometric particularities of it.

Case of a perpendicular axis

The dimension of the ellipse is characterized by the values of its semi-axis, $a$ and $b$. A point $M$ placed on it will be at a distance $x$ of the ellipse centre (fig. 5). This distance can be calculated starting from the ellipse equation and the point coordinates:

$$\left\{ \begin{array}{c} x_M = x \cos \theta \, , \\
y_M = x \sin \theta \, \end{array} \right.$$ 

$$\frac{x_M^2}{a^2} + \frac{y_M^2}{b^2} - 1 = 0 \quad \Rightarrow x = \frac{b^2}{\sqrt{(b/a)^2 + \sin^2 \theta \cdot (1 - (b/a)^2)}}. \quad (6)$$

In order to determine the magnetic field strength produced by the elliptic coil one starts from the Biot-Savart-Laplace’s formula. The elementary section of the elliptic coil passed by the current $I$ is estimated using the elliptic integral of second kind [10], [11]:

$$dr = a \sqrt{1 - \left(1 - \frac{b^2}{a^2}\right) \sin^2 \theta} \cdot d\theta \quad (7)$$

Thus the magnetic field strength in any point on this line will be oriented along this axis and will be calculated with formula:

$$H = H_x = \frac{I}{4\pi} \int_0^{2\pi} x a \sqrt{1 - \left(1 - \frac{b^2}{a^2}\right) \sin^2 \theta} \cdot d\theta \quad (8)$$

Also in this case a numerical integration method will be used for the estimation of the values of $H$ given by relation (8), creating a program in MATLAB. Some of the obtained results are depicted in fig. 6, for a circle ($a = b$) and for an ellipse. One observes that, for the circle case, the results are similar to those of fig. 1 (a).
Case of an axis inclined with a given angle

The approach of this case is based on the techniques presented in the previous cases. Following a similar principle of calculation that starts from Biot-Savart-Laplace’s formula, and processing the results obtained for a circular coil and an axis inclined with an angle $\alpha$ [12], we obtained the following expressions that approximate the components of the field, along the axis and respectively perpendicular on it (fig.7):

\[
H_x = \frac{I \cdot a \cdot \sin \alpha}{4\pi} \cdot F(\theta),
\]

\[
F(\theta) = \int_0^\pi \sin \gamma \sqrt{1 - \left(1 - \left(\frac{b}{a}\right)^2\right)^2} \sin^2 \theta \left(\frac{1}{(x^2 + h^2 + 2xh \cos \alpha)^{3/2}} + \frac{1}{(x^2 + h^2 - 2xh \cos \alpha)^{3/2}}\right) d\theta;
\]

\[
H_y = \frac{I \cdot a}{4\pi} \cdot G(\theta),
\]

\[
G(\theta) = \int_0^\pi \cos \gamma \sqrt{1 - \left(1 - \left(\frac{b}{a}\right)^2\right)^2} \sin^2 \theta \left(\frac{|h - x \cos \alpha|}{(x^2 + h^2 + 2xh \cos \alpha)^{3/2}} - \frac{|h + x \cos \alpha|}{(x^2 + h^2 - 2xh \cos \alpha)^{3/2}}\right) d\theta,
\]

where $\gamma$, the angle between the vectors $\vec{d}r$ and $\vec{R}$, was approximated with the relation

\[
\gamma \cong \frac{\pi}{2} + \cos \alpha \cdot \left(\frac{2}{\pi} \alpha - 1\right) \cdot \theta.
\]
Figures 8 (a) and (b) present some of the results obtained by implementation of the above relations in a MATLAB program. Figure 8 (a) depicts the variation curves of $H_x$ (along the axis) with the distance to the ellipse centre, for a circle ($a = b$) and for an ellipse. The results obtained for the circle are similar to those obtained using the formulae (4)-(5) from the previous paragraph. Figure 8 (b) shows the variation of $H$ along the axis, for different angles of inclination of the line.

4 Conclusion

From the analysis we made results that it is extremely difficult to obtain the exact formulae for the magnetic field strength, especially in case of an inclined straight line. The only solution in these cases is to use some numerical integration techniques that approximate the obtained integral formulae.

The study proposed and verified using alternative procedures could represent a more available solution in order to estimate the magnetic field, when we do not have a specialized program for the electromagnetic field modelling in 3D, rather expensive, necessary for geometric forms that do not have planar of axial symmetries.

This analysis could be extended afterwards to the coils with several turns or for other kind of shapes that include circular or elliptical sections.

This study is useful in the engineering practice, especially for the finite length coils' case, having the radius much bigger than their length or a reduced number of turns (as in case of the coils used in the wireless transfer of energy [13]).

References